8.2 Polar Equations

Just as we've used equations in x and y to represent relations in rectangular coordinates, equations in the variables r and θ represent relations in polar coordinates. We use **Theorem 8.1** to convert equations between the two systems.

Converting from Rectangular to Polar Coordinates

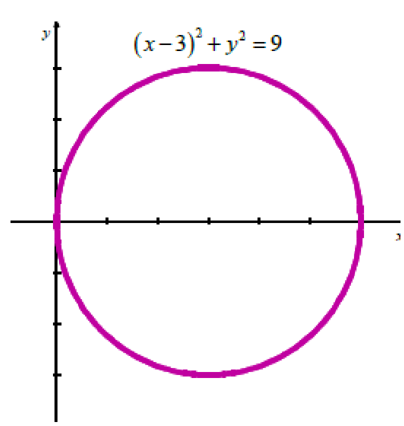
One strategy to convert an equation from rectangular to polar coordinates is to replace every occurrence of x with $r\cos(\theta)$ and every occurrence of y with $r\sin(\theta)$, and use identities to simplify. This is the technique we employ in the following three examples.

Example 8.2.1. Convert $(x-3)^2+y^2=9$ from an equation in rectangular coordinates into an equation in polar coordinates.

Solution. We start by substituting $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $(x-3)^2+y^2=9$ and then simplify. With no real direction in which to proceed, we follow our mathematical instincts and see where they take us.[1]

$$
\begin{aligned}
(x-3)^2+y^2&=9\\
(r\cos(\theta)-3)^2+(r\sin(\theta))^2&=9\\
r^2\cos^2(\theta)-6r\cos(\theta)+9+r^2\sin^2(\theta)&=9\\
r^2\cos^2(\theta)+r^2\sin^2(\theta)-6r\cos(\theta)&=0 \quad \text{after subtracting 9 from both sides}\\
r^2\left(\cos^2(\theta)+\sin^2(\theta)\right)-6r\cos(\theta)&=0\\
r^2-6r\cos(\theta)&=0 \quad \text{since } \cos^2(\theta)+\sin^2(\theta)=1\\
r(r-6\cos(\theta))&=0 \quad \text{after factoring}
\end{aligned}
$$

We get $r=0$ or $r=6\cos(\theta)$. Recognizing the equation $(x-3)^2+y^2=9$ as describing a circle, we exclude the first since $r=0$ describes only a point (namely the pole/origin). We choose $r=6\cos(\theta)$ for our final answer.



Note that when we substitute $\theta=\dfrac{\pi}{2}$ into $r=6\cos(\theta)$, we recover the point $r=0$, so we aren't losing anything by disregarding $r=0$.

8.2 Polar Equations

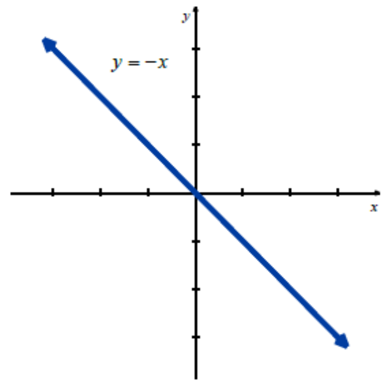
Example 8.2.2. Convert $y=-x$ from an equation in rectangular coordinates into an equation in polar coordinates.

Solution. We substitute $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $y=-x$.

$$\begin{aligned} y &= -x \\ r\sin(\theta) &= -r\cos(\theta) \\ r\cos(\theta)+r\sin(\theta) &= 0 && \text{after rearranging} \\ r\left(\cos(\theta)+\sin(\theta)\right) &= 0 && \text{after factoring} \end{aligned}$$

This gives $r=0$ or $\cos(\theta)+\sin(\theta)=0$. Solving the latter for θ, we get $\theta=-\frac{\pi}{4}+\pi k$ for integers k.

As we did in the previous example, we take a step back and think geometrically. We know $y=-x$ describes a line through the origin. As before, $r=0$ describes the origin but nothing else. Consider the equation $\theta=-\frac{\pi}{4}$. In this equation, the variable r is free, meaning it can assume any and all values including $r=0$. If we imagine plotting points $\left(r,-\frac{\pi}{4}\right)$ for all conceivable values of r (positive, negative and zero), we are essentially drawing the line containing the terminal side of $\theta=-\frac{\pi}{4}$ which is none other than $y=-x$.



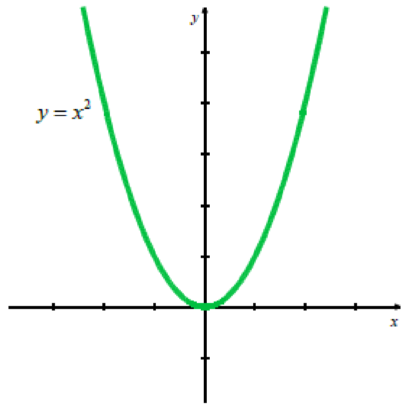
Hence, we can take as our final answer $\theta=-\frac{\pi}{4}$.

Example 8.2.3. Convert $y=x^2$ from an equation in rectangular coordinates into an equation in polar coordinates.

Solution. We substitute $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $y=x^2$.

$$\begin{aligned} y &= x^2 \\ r\sin(\theta) &= \left(r\cos(\theta)\right)^2 \\ r\sin(\theta) &= r^2\cos^2(\theta) \\ 0 &= r^2\cos^2(\theta)-r\sin(\theta) \\ 0 &= r\left(r\cos^2(\theta)-\sin(\theta)\right) \end{aligned}$$

8.2 Polar Equations



Either $r = 0$ or $r\cos^2(\theta) = \sin(\theta)$. We can solve the latter equation for r by dividing both sides of the equation by $\cos^2(\theta)$.

As a general rule we never divide through by a quantity that may be equal to 0. In this particular case, we are safe since if $\cos^2(\theta) = 0$ then $\cos(\theta) = 0$ and, for the equation $r\cos^2(\theta) = \sin(\theta)$ to hold, then $\sin(\theta)$ would also have to be 0. Since there are no angles with both $\cos(\theta) = 0$ and $\sin(\theta) = 0$, we are not losing any information by dividing both sides of $r\cos^2(\theta) = \sin(\theta)$ by $\cos^2(\theta)$. Doing so, we get

$$\begin{aligned} r &= \frac{\sin(\theta)}{\cos^2(\theta)} \\ &= \frac{1}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)} \\ &= \sec(\theta)\tan(\theta) \end{aligned}$$

As before, the $r = 0$ case is recovered in the solution $r = \sec(\theta)\tan(\theta)$ when $\theta = 0$. So we state our final solution as $r = \sec(\theta)\tan(\theta)$.

□

Converting from Polar to Rectangular Coordinates

As a general rule, converting equations from polar to rectangular coordinates isn't as straight forward as the reverse process. We will begin with the strategy of rearranging the given polar equations so that the expressions $r^2 = x^2 + y^2$, $r\cos(\theta) = x$, $r\sin(\theta) = y$ and/or $\tan(\theta) = \frac{y}{x}$ present themselves.

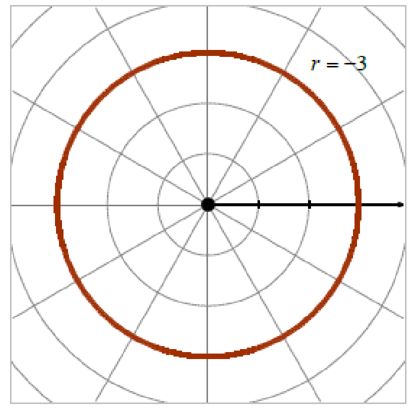
Example 8.2.4. Convert $r = -3$ from an equation in polar coordinates into an equation in rectangular coordinates.

Solution. Starting with $r = -3$, we can square both sides.

$$\begin{aligned} r &= -3 \\ r^2 &= (-3)^2 \\ r^2 &= 9 \end{aligned}$$

8.2 Polar Equations

We may now substitute $r^2 = x^2 + y^2$ to get the equation $x^2 + y^2 = 9$.



As we have seen, squaring an equation does not, in general, produce an equivalent equation. The concern here is that the equation $r^2 = 9$ might be satisfied by more points than $r = -3$. On the surface, this appears to be the case since $r^2 = 9$ is equivalent to $r = \pm 3$, not just $r = -3$. However, any point with polar coordinates $(3, \theta)$ can be represented as $(-3, \theta + \pi)$, which means any point (r, θ) whose polar coordinates satisfy the relation $r = \pm 3$ has an equivalent[3] representation which satisfies $r = -3$.

Thus, we state our final solution as $x^2 + y^2 = 9$.

□

Example 8.2.5. Convert $\theta = \frac{4\pi}{3}$ from an equation in polar coordinates into an equation in rectangular coordinates.

Solution. We begin by taking the tangent of both sides of the equation.

$$\theta = \frac{4\pi}{3}$$

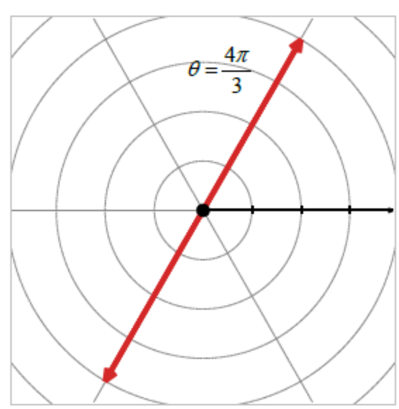
$$\tan(\theta) = \tan\left(\frac{4\pi}{3}\right)$$

$$\tan(\theta) = \sqrt{3}$$

Since $\tan(\theta) = \frac{y}{x}$, we get the following.

$$\frac{y}{x} = \sqrt{3}$$

$$y = x\sqrt{3}$$



Of course, we pause a moment to wonder if, geometrically, the equations $\theta = \frac{4\pi}{3}$ and $y = x\sqrt{3}$ generate the same set of points.[4]

The same argument presented in **Example 8.2.4** applies equally well here.

We conclude that our answer of $y = x\sqrt{3}$ is correct.

□

8.2 Polar Equations

Example 8.2.6. Convert $r = 1 - \cos(\theta)$ from an equation in polar coordinates into an equation in rectangular coordinates.

Solution. Once again, we need to manipulate $r = 1 - \cos(\theta)$ a bit before using the conversion formulas given in **Theorem 8.1**. We could square both sides of this equation like we did in **Example 8.2.4** to

obtain an r^2 on the left hand side, but that does nothing helpful for the right hand side. Instead, we multiply both sides by r and continue manipulating the equation so that we can apply the conversion formulas from **Theorem 8.1**.

$$
\begin{aligned}
r &= 1 - \cos(\theta) \\
r^2 &= r - r\cos(\theta) && \text{multiplying through by r} \\
r^2 + r\cos(\theta) &= r && \text{adding } r\cos(\theta) \text{ to both sides} \\
\left(r^2 + r\cos(\theta)\right)^2 &= r^2 && \text{squaring both sides} \\
\left(x^2 + y^2 + x\right)^2 &= x^2 + y^2 && \text{substituting } r^2 = x^2 + y^2 \text{ and } r\cos(\theta) = x
\end{aligned}
$$

In the last step, we applied **Theorem 8.1** and we now have the equation $\left(x^2 + y^2 + x\right)^2 = x^2 + y^2$ as a solution.

Last Example:

Convert equation from polar coordinates to rectangular coordinates.

$r = -2\sec(\theta)$ hint: rewrite secant in terms of cosine, multiply both sides of the equation by cosine